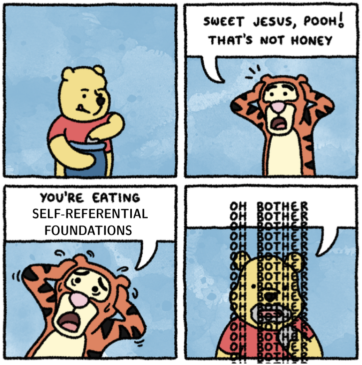
“Proofs *are* programs” in MLTT



Pierre-Marie Pédrot

INRIA

TYPES'24

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Beyond obvious.

- Proofs are first-class syntactic objects
- Extension of the λ-calculus
- The equational theory can be derived from β-reduction
- Strong normalization and canonicity
- No need for *post-hoc* realizability

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MLTT: The Ultimate Synthetic Monistic Curry-Howard System

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On second thought, it is not so clear.

- MLTT has uncomputational models
- For instance, in $\mathbf{Set}$ functions are ZFC functional graphs
- Is our Curry-Howard faith grounded in reality?

A Hint of Heresy

Our credo is just an **external** statement!

Proofs *are* programs in MLTT

The Problem

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What we really want is an **internal** statement.

"Proofs *are* programs" in MLTT

Church's Church

Turns out it is a well-known principle in constructive maths.

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The internal Church Thesis

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where $\mathbf{calc}\ f\ p$ means that f is computed by the program p, i.e.

$$\vdash \forall n : \mathbb{N}.\, \exists k : \mathbb{N}.\, \mathtt{eval}\ p\ n\ (f\ n)\ k$$

with $\mathtt{eval} : \mathbb{N} \to \mathbb{N} \to \mathbb{N} \to \mathbb{N} \to \square$ the Kleene predicate

"$\mathtt{eval}\ p\ n\ v\ k \quad \sim \quad$ the Turing machine p run on n returns v in $\leq k$ steps."

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[For readability, I'll henceforth write $\mathbb{P} := \mathbb{N}$ to indicate numbers coding programs]

I am not Making this CT Up

Synthetic Computability

Never suffer with Turing machines again!

Satisfied or money back

AS SEEN ON TV!

Prove computability results (almost) pain-free in Coq!

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The one missing primitive: inspecting the code of a program, a.k.a. CT.

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"Can we extend Martin-Löf's Type Theory with CT?"

I think, Therefore I merely am

In dependent type theories, existing is a complex matter

$\Sigma x : A.\, B$	v.s.	$\exists x : A.\, B$
actual existence		mere existence
proof relevant		proof-irrelevant
choice built-in		no choice *a priori*
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We have not one, but *two* theses.

$$\begin{array}{lcl} \mathsf{CT}_{\exists} & := & \Pi(f : \mathbb{N} \to \mathbb{N}).\, \exists p : \mathbb{P}.\, \mathbf{calc}\ f\ p \\ \mathsf{CT}_{\Sigma} & := & \Pi(f : \mathbb{N} \to \mathbb{N}).\, \Sigma p : \mathbb{P}.\, \mathbf{calc}\ f\ p \end{array}$$

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Which do we want?

I choose you, Sigma-chu

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- $\mathsf{MLTT} + \mathsf{CT}_\exists$ is known to be consistent

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- Intuitionistic non-choice gives a quote function $(\mathbb{N} \to \mathbb{N}) \to \mathbb{P}$
- Consistency of $\mathsf{MLTT} + \mathsf{CT}_\Sigma$ is not established

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This is the one we really want to have in MLTT!

Second-hand "Quotes" from Anonymous Experts**

M.E. (Birmingham)

"MLTT is obviously inconsistent with CT_Σ"

"I believe that MLTT cannot validate CT_Σ"

T.S. (Darmstadt)

** All these quotes are a pure work of fiction. Serving suggestion. May contain phthalates.

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But consistency of $\mathsf{MLTT} + \mathsf{CT}_\Sigma$ is *obviously* trivial...

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Only one way out: prove that I am right!

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Spoiler alert: we will sketch that in the rest of the talk.

“MLTT”

“MLTT” is the extension of MLTT with three quoting primitives.

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These three operations are just the Skolemization of CT_Σ!

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Convert Now or Face Type Shunning

Where is the type-theoretic fish?

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- A meta-function $\lceil \cdot \rceil : \texttt{term} \Rightarrow \mathbf{N}$ (your favourite Gödel numbering)

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- An MLTT function $\vdash \mathtt{run} : \mathbb{P} \to \mathbb{N} \to \mathfrak{P}(\mathbb{N})$

where $\mathfrak{P}(A) := \mathbb{N} \to 1 + A$ is the partiality monad and `eval` is derived from `run`

Wake Up Sheeple!

Open terms are a lie! It's a conspiracy from Big Variable!

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(Congruences trivial, similar rule for ϱ.)

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This One Weird Trick

Closed terms are stable by substitution.

(Some additional technicalities to validate η-laws.)

The Basic Model

A straightforward variant of Abel's style NbE logical relation

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⇝ annotate reducibility proofs with deep normalization

$$\Gamma \Vdash M : A \qquad \text{implies} \qquad M \Downarrow_{\mathsf{deep}} M_0 \quad \text{with} \quad \Gamma \vdash M \equiv M_0 : A$$

⇝ normal / neutral terms generalized into deep and weak-head variants

⇝ extend neutrals to contain quoting primitives blocked on open terms

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... and that's about it.

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- Weak-head reduction of quoting primitives depends on deep reduction

$$\frac{M \Rightarrow_{\mathsf{deep}} R}{\text{♀}\ M \Rightarrow_{\mathsf{wh}} \text{♀}\ R} \qquad \frac{M \text{ closed dnf}}{\text{♀}\ M \Rightarrow_{\mathsf{wh}} \lceil M \rceil}$$

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Better presented as a step-indexed big-step reduction

$$M \Rightarrow^{*}_{\mathsf{deep}} N\ \text{dnf} \qquad \text{iff} \qquad \exists k.\ M \Downarrow^{k} N$$

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We say that the computation model $(\lceil\cdot\rceil, \texttt{run})$ is adequate when:
for all $M \in \texttt{term}$ and $n, r, k \in \mathbf{N}$, $M\,\overline{n} \Downarrow^{k}\ \overline{r}$ implies

- $\texttt{run}\ \lceil M \rceil\ \overline{n}\ \overline{k} \Downarrow \mathsf{Some}\ \overline{r}$
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- $\mathtt{run}\ \lceil M \rceil\ \overline{n}\ \overline{k} \Downarrow$ Some $\overline{r}$
- $\mathtt{run}\ \lceil M \rceil\ \overline{n}\ \overline{k'} \Downarrow$ None for all $k' < k$

Theorem

If the model is adequate, the logical relation is sound and complete.

The Real Results

Theorem (It's written on the can)

"MLTT" proves CT_Σ*.*

Theorem (Consistency)

There is no closed term of type $\perp$ *in "MLTT".*

Theorem (Canonicity)

All closed terms of type $\mathbb{N}$ *in "MLTT" reduce to an integer.*

Theorem (Normalization)

Well-typed "MLTT" terms are strongly normalizing.

Formalization

Based on Adjedj et al. CPP'24 "Martin-Löf à la Coq" (using small IR)

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Fully formalized in Coq up to one axiom.

Nightmare stuff I'm not gonna prove: the existence of adequate models

Typical instance of "conceptually trivial but practically impossible".

We have *already* implemented an adequate model in the Coq meta.

MLTT contains PRA and the evaluator is primitive recursive.

Conclusion

In MLTT, "proofs *are* programs" in the end

- The model is a trivial adaptation of standard NbE models
- Essentially fully formalized in Coq
- Open terms do not exist, I have met them
- What kind of axioms can we cheaply internalize like this?

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A lingering doubt

Why was this considered a difficult question?

Thanks for your attention.